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TWO ITEMS CONCERNING DIRECTIONAL DATA

BY

M. A. STEPHENS

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1. A discussion is given of confidence procedures for the model vector $\underset{\sim}{A}$ of a von Mises or Fisher distribution. Slight revisions of confidence levels are suggested because the usual procedure strictly gives the axis but not the direction of $\underset{\sim}{A}$.

2. The likelihood ratio test is given, with percentage points, for testing $\underset{\sim}{A} = \underset{\sim}{A}_0$, for the Fisher distribution. A good approximation is offered for the similar test for the von Mises distribution.

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TWO ITEMS CONCERNING DIRECTIONAL DATA

by

M.A. Stephens

In this paper we consider two topics connected with directional data, i.e., observations which may be recorded by unit vectors $\underset{\sim}{OP}_i$ from the center O of a circle or sphere of radius 1, to points P_i on the circumference or surface. Alternatively, the point P_i on a circle may record an observation, such as the occurrence of an event during a period whose length is represented by the total circumference.

When the observations are clustered around a central direction, the von Mises distribution, on the circle, or the Fisher distribution, on the sphere, are used to describe the data. These are unimodal probability distributions, with density on the surface proportional to $\exp(\kappa\cos\alpha)$, where κ is a concentration parameter and α the angle between $\underset{\sim}{OP}$ and the central (modal) vector $\underset{\sim}{A}$. It will be assumed that the reader is familiar with these distributions; the references given in the text may be used as sources for earlier references.

The two topics to be discussed in the next two sections are: In section 2; A discussion of the standard tests and confidence interval procedures for the modal vector $\underset{\sim}{A}$ of a von Mises or a Fisher distribution, when the concentration paraketer κ is not known. It is not generally noted that the usual test is strictly for the axis along which $\underset{\sim}{A}$ lies, without specifying the direction, and this necessitates a slight revision of confidence levels.

In section 3, Significance points are provided for the likelihood-ratio test for $\underset{\sim}{A}$ for a Fisher distribution, for the case when the concentration parameter κ __is__ known. The test will replace a test already suggested elsewhere, and indications are that an approximation will hold well for the von Mises distribution also.

2. __Tests and Confidence Intervals for the Modal Vector of a von Mises or Fisher Distribution.__

2.1. The von Mises and Fisher distributions have densities proportional to $\exp(\kappa\cos\alpha)$, where α is the angle between a sample vector $\underset{\sim}{OP}$ and the __modal vector__ $\underset{\sim}{A}$, and κ is a concentration parameter. For a test of H_o: $\underset{\sim}{A}$ is along a given $\underset{\sim}{A}_o$, when κ is not known, Watson and Williams (1956) proposed a conditional test, both for the circle and the sphere. Suppose a sample of N unit vectors gives a resultant vector $\underset{\sim}{R}$, length R, and let its component on $\underset{\sim}{A}_o$ be X; the test depends on the fact that the conditional distribution of R, given X, is independent of κ. Critical values R_o, for given N, X, and α, have been given by Stephens (1962a,b) in the form of charts of R_o against X; if R exceeds R_o, the null hypothesis is rejected.

The distribution of R, given X, is in fact the same for X positive or negative, and if a given $\underset{\sim}{A}_o$ is acceptable when the component of $\underset{\sim}{R}$ on $\underset{\sim}{A}_o$ is positive, the vector $-\underset{\sim}{A}_o$ would be equally acceptable, the component being now negative. Thus the test is strictly a test for the modal axis, without direction. This will become important when we use critical values R_o of R to obtain a confidence interval for $\underset{\sim}{A}$. The use of the charts both for testing and for confidence

intervals, can best be illustrated with the help of a diagram. In Figure 1 we have taken the case of a sample of size $N = 20$, drawn from the von Mises distribution.

Suppose $\underset{\sim}{OX}$ points in the direction of the modal vector $\underset{\sim}{A}$, and let OY be the axis at right angles. For every sample with resultant $\underset{\sim}{R}$, we can calculate $Y = \pm\sqrt{R^2 - X^2}$, and set $C \equiv (X,Y)$. $\underset{\sim}{OC}$ is the resultant $\underset{\sim}{R}$, and C represents the sample on the diagram. C lies inside the circle, center O, radius 20. Let R_0 be the critical value of R for given X, at the 5% level, given by the charts, and draw the curve $Y^2 = R_0^2 - X^2$; call this the <u>5% critical limit</u>. This is shown in the illustration, and also part of the 1% critical limit obtained from 1% critical values of R. The curves are symmetrical about OY as well as OX. Since for every X the probability of C falling outside the curve shown in 0.05, with Y either positive or negative, the probability of falling <u>above the upper curve</u> is 0.025 and of falling below the lower curve is also 0.025, <u>whatever the value of κ</u>. Suppose now OX has coordinate θ_0 measured anticlockwise from a suitable initial line OB; if the true modal vector were along OX' with angle $\theta_1 > \theta_0$, more than 2.5% of sample points C would fall above the upper 2.5% curve. In the same way, if $\theta_1 < \theta_0$ an excess of sample points will lie below the lower 2.5% critical boundary. As a result, a test of the hypothesis, H_0, that the direction of the modal vector is $\theta_c = \theta_0$, against the one-sided alternative that $\theta_c > \theta_0$, consists in (a) Calculating X and R from the sample, (b) finding from the charts in Stephens (1962a) the critical value R_0 of $R(.05)|X$ (or $R(.01|X)$) and (c) rejecting H_0 at the

2.5%(or 0.5%) level, if the observed $R > R_o$. There will be a corresponding one-sided test against the alternative $\theta_c < \theta_o$, and if the alternative is simply $\theta_c \neq \theta_o$, then the steps above are followed but, in (c), H_o is rejected, if R is too large, at the 5% (or 1%) level. For N, X not given in the charts, Stephens (1962a) has given several approximations. Similar charts and approximations for use with the Fisher distribution, are given in Stephens (1962b).

2.2 Confidence limits for θ_c.

Suppose that the true modal vector has direction $\theta_c = \theta_o$, but that we are unaware of this and must estimate θ_c from the direction of the sample vector resultant. Figure 1 shows two possible resultants OC_1, OC_2, at angles θ_1, θ_2 respectively, both of the same length $R = 7.90$. We find from Stephens' tables that for $N = 20$ the corresponding abscissa for $\alpha = 0.05$, is $X = 5.0$, and note that

$$\varphi = \cos^{-1}(X|R) = \cos^{-1} = \cos^{-1}(5.0/7.9) = \cos^{-1}(.6329) = 50.7^\circ.$$

φ is clearly the angle between OZ and OX, where OZ is 7.9 and Z is on the 5% critical limit.

Suppose the lower confidence limit for θ_c is obtained by subtracting φ from the angle of $\underset{\sim}{R}$; the two illustrations then give limits θ_1-50.7° and θ_2-50.7°. In the first example this limit gives a confidence interval which does not include the true modal vector, along OX, and this is clearly because the point C_1 is above the 5% critical boundary; while the interval based on OC_2 does include OX, because C_2 is below the 5% boundary. Since 2.5% of samples

would give a sample point above the boundary, like C_1, and 97.5% below, like C_2, the procedure clearly provides a lower 97.5% confidence limit for θ_c, whatever is the value of κ. Similarly, if we add $\varphi = 50.7^\circ$ to the angle of the sample resultant vector we shall have an upper 97.5% confidence limit for θ_c, and the two limits will define a central 95% confidence interval for θ_c. A similar result holds for the left side of the diagram, where X is negative. Thus the interval is not strictly for the modal vector with direction, but only for the axis along which it lies, with either possible direction. In practice, of course, except for κ very small, negative values of X will rarely occur, and the confidence interval is always chosen to give positive X, but this lowers the confidence level. The amount by which it is lowered is found as follows. Let p be the probability, for given κ, that $X > 0$, i.e., the probability that the sample point falls to the right of OY. If P is the confidence probability (i.e., $100P$% is the confidence level) then the probability that the above procedure includes the positive modal vector is Pp. The relationship of p and κ, for p near 1, may be found approximately from a table of percentage points of X, in Stephens (1969a). From this we have

N:	10	10	20	20	40	40
p:	0.95	0.99	0.95	0.99	0.95	0.99
κ:	0.74	1.08	0.53	0.74	0.37	0.55

Thus for $\kappa > 1$, the risk of mistaking the direction of $\underline{A}$ will be very small, for $N = 10$ say, and the confidence level hardly changes;

for larger samples, κ may become even smaller with negligible risk of a wrong decision.

2.3 The Fisher Distribution.

For three-dimensions, the same type of argument holds, though a geometric representation would now involve an ellipsoid for the 5% critical limit. Corresponding to two-sided confidence intervals for θ on the circle, there will now be a cone of confidence for θ on the sphere; it would be difficult to interpret one-sided tests or confidence intervals. Again the test based on R for given X is really a test for the axis of $\underset{\sim}{A}$, without direction, and the alleged confidence probability P must be multiplied by a p which depends on N and κ. From tables of the distribution of X for given N, κ given in Stephens (1967) the table of values for p for the sphere becomes

N:	10	10	20	20	40	40
p:	0.95	0.99	0.95	0.99	0.95	0.99
κ:	0.91	1.30	0.64	0.91	0.45	0.64

κ is required to be a little higher for the sphere than for the circle to obtain the same p for given N.

2.4 Numerical illustration.

In Table 1 we give, for N = 10 and 20, the relationship between R/N and the critical value X/N for a 95% confidence interval with the Fisher distribution; also the θ_c to which this corresponds.

Further, R/N is used to estimate κ (by $\hat{\kappa}$, the solution of $\coth\kappa - 1 - \kappa = R/N$); the estimate could then be used to estimate p as described above. Values of $\hat{\kappa}$ are also included in Table 1.

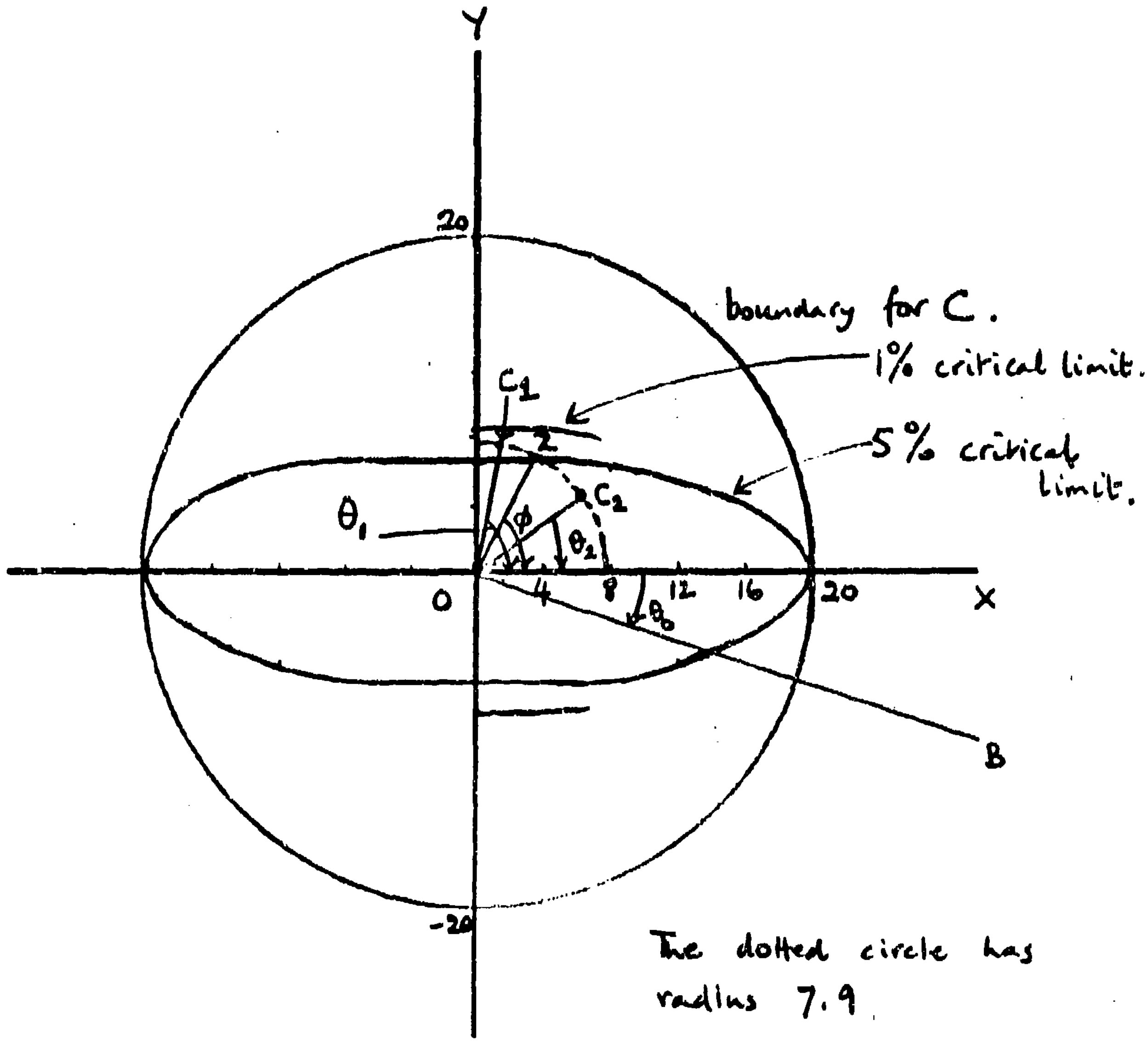
Y
20
boundary for C.
1% critical limit.
5% critical limit.
C_1
2
C_2
θ_1
ϕ
θ_2
0
4
8
12
16
20
X
θ_0
B
-20
The dotted circle has radius 7.9

Fig 1. N = 20

<u>TABLE 1</u>

Critical values of X/N and θ_c (to nearest degree) for 95% confidence interval, given R/N, and estimated value $\hat{\kappa}$ derived from R/N. (Fisher distribution).

N = 10

R/N:	.44	.45	.5	.55	.6	.7	.8	.9
X/N:	.00	.056	.241	.342	.431	.582	.721	.861
θ_c:	90	83	61	52	44	34	26	17
$\hat{\kappa}$:	1.51	1.55	1.80	2.07	2.40	3.30	5.00	10.00

N = 20

R/N:	.32	.35	.4	.5	.6	.7	.8	.9
X/N:	.00	.16	.29	.42	.53	.65	.77	.88
θ_c:	90	63	44	34	28	22	17	11
$\hat{\kappa}$:	1.03	1.14	1.34	1.80	2.40	3.30	5.00	10.00

3. Likelihood-Ratio Test for the Modal Vector of a Fisher Distribution When the Concentration Parameter is Known.

3.1 The test to be discussed in this section is similar to that in Section 2, but with the concentration parameter κ assumed known. This is probably less likely to occur, but when it does, a more powerful test of H_o can be made. The likelihood-ratio test statistic for H_o: $\underset{\sim}{A}$ along $\underset{\sim}{A}_o$, against the alternative H_A: that $\underset{\sim}{A}$ is along a vector other than $\underset{\sim}{A}_o$, may be shown to be R-X. Thus if the distribution of Z = R-X were known, a test based on Z, rejecting if Z were too large, should give a more powerful test against the alternative. The exact distribution of Z is difficult to find; however, for the Fisher distribution, significance points can be found, and are given in Table 2. The test may formally be set out as follows:

(a) Calculate Z = R-X;

(b) In Table 2 for given N, κ and α, find the table entry z;

(c) If $Z > z$, reject H_o at significance level α.

If κ is too large for Table 2, solve for z from

$$2\kappa z = \chi_2^2(\alpha) , \tag{1}$$

where $\chi_2^2(\alpha)$ is the upper significance point of χ_2^2 at level α.

3.2 Theory of the test.

We start by defining

$$c(\kappa) = \left(\frac{\kappa}{2\sinh\kappa}\right)^N \text{ , and}$$

$$P_N^r(t) = \sum_{s=0}^{\infty} \binom{N}{s}(-1)^s < N-t-2s >^r \text{ ,}$$

N and r are positive integers, and $|t| < N$; the notation $< z >$ means $< z > = z$ if $z > 0$, and $< z > = 0$ if $z \le 0$. The joint density of R and X is (Stephens, 1967)

$$f_1(R,X) = c(\kappa)\exp(\kappa X)P_N^{N-2}(R)/(N-2)! \quad , \quad |X| < R,\ 0 < R < N; \qquad (2)$$

Let $y = R-X$; the joint density of y, R is then

$$f_2(y,R) = c(\kappa)\exp(-\kappa y)\exp(\kappa R)P_N^{N-2}(R)/(N-2)! \quad ;$$

$$0 \le y \le 2R \text{ , } 0 \le R \le N \qquad (3)$$

For the density of y alone, we must integrate out R, so we require

$$J = \int_V^N \exp(\kappa R)P_N^{N-2}(R)dR \text{ , with } V = y/2 \text{ .}$$

This integral J enters an expression for an integral I given in Stephens (1967, top of page 216) and may easily be deduced from the result for I; the density of y then becomes

$$f_3(y) = \kappa c(\kappa)\exp(-\kappa y)\left[\exp(-\kappa y/2)\sum_{r=2}^{N-1}\frac{P_N^{N-r}(y/2)}{\kappa^r (N-r)!} + \frac{1}{\kappa^N}\sum_{r=0}^{t}(\exp(\kappa(N-2r))-\exp(\kappa y/2))\binom{N}{r}(-1)^r\right], \quad 0 \le y \le 2N .$$

Where t is the greatest integer less than $(N-y/2)/2$. This density has been integrated numerically to give the significance points in Table 2.

The χ^2 approximation in step (c) of Section 3.2 may be deduced from the fact that $-2 \ln \lambda = 2\kappa(R-X)$, where λ is the likelihood ratio for the test of H_0, and so is asymptotically χ^2 distributed. However, Watson (1956) first gave the approximation, and deduced it from the properties of the Fisher density for large κ. It is interesting also to see that it follows from (2) above, if κ is large, since in this circumstance we expect large R and X and therefore small values of y. Thus if we replace the range of y in (2) by $0 \le y \le \infty$, and adjust the constant $c(\kappa)$ accordingly, we can see that the joint density of y and R now factors into two components, one containing only y and the other only R; with appropriate choice of constant terms, these must be the marginal densities of y and R. For y, since the term with y is $\exp(-\kappa y)$, we see that κy must have the exponential distribution; the full density is

$$f(y) = \kappa\exp(-\kappa y) , \quad y > 0$$

and this is equivalent to

$$2\kappa y \simeq \chi_2^2 . \tag{3}$$

These two approximate derivations of (3) show that we can expect it to be a good approximation: (a) when κ is large, for all N, and (b) for smaller values of κ, as $N \to \infty$. The values given by the approximation are recorded in Table 2 opposite $N = \infty$; the exact values given for other values of κ show that the above expectations are borne out extremely well.

3.3 Power comparisons.

The test of H_0 above can also be made using X alone, since its distribution for given N and κ is known and significance points tabulated. This suggestion was given in Stephens (1967), but was wrongly described as a two-tailed test, in fact, H_0 would be rejected only if X is too small for given N and κ. We now compare the two tests for power, not directly, but indirectly by comparing the confidence intervals for $\underset{\sim}{A}$ obtained by the two methods. Suppose $\underset{\sim}{A}$ is at angle θ to $\underset{\sim}{R}$; a $(1-\alpha)\%$ confidence interval for θ is $0 \leq \theta \leq \theta_c$, where θ_c is the solution of $R \cos \theta_c = X_c$, and X_c is the critical value of X found in either test, at significance level α.

Consider an example, when $N = 10$ and $\kappa = 4$ The 5% critical value of $R-X$ is found from step (c), the value is 0.750, so $X_c = R-0.750$. For the test based on X alone, $X_c = 6.10$ (from Table 1, Stephens (1967)). Thus the $R-X$ test will give smaller confidence band for θ whenever $R-0.750 > 6.10$, i e., whenever $R > 6.85 = R_0$ say. From Table 2 of Stephens (1967) the probability $\Pr(R > R_0)$ can roughly be estimated; for $N = 10$, $\kappa = 4$, the lower 1% and 5% values

of R are 6.42 and 5.71 and $\Pr(R > 6.85)$ is near 0.9. Thus 90% of the time the R-X test, for $N = 10$, $\kappa = 4$, would give greater power than the test based on X alone. For $N = 10$, and other values of κ, the value of R_o is given, and a rough estimate of the probability p of exceeding it, in the following small table:

κ:	1	1.5	2	2.5	3	4
R_o:	3.21	3.78	4.56	5.29	5.92	6.85
p:	0.5	0.7	0.75	0.85	0.88	0.90

p represents the probability that the Z-test is better than the X-test. The same pattern is repeated if one makes the calculations for $N = 20$. Clearly the Z-test is on the whole a better test.

3.4 Corresponding test for the circle.

For the von Mises distribution also, $Z = R-X$ is the likelihood-ratio test statistic for H_o; the exact distribution cannot be handled, but the asymptotic result is $2\kappa(R-X) \simeq \chi_1^2$, so that critical values z of Z would be given by

$$2\kappa z = \chi_1^2(\alpha) \quad .$$

Monte Carlo results suggest that this approximation will hold with in a manner similar to that for the sphere, i.e., for $\kappa \geq 2$, and $N \geq 20$, and, for larger κ, even for $N \leq 20$.

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TABLE 2. Critical Values of R-X, for Test of Modal Vector, Sphere.

In descending order the values are for α = .01, .025, .05, .10

N \ k	0.5	1.0	1.5	2.0	2.5	3.0
4	4.722	3.746	2.902	2.273	1.837	1.534
	4.057	3.113	2.357	1.828	1.473	1.229
	3.486	2.597	1.931	1.487	1.197	0.998
	2.847	2.047	1.495	1.145	0.920	0.767
5	5.161	3.950	2.969	2.289	1.840	1.535
	4.407	3.258	2.399	1.837	1.475	1.229
	3.767	2.701	1.959	1.493	1.198	0.999
	3.06	2.117	1.512	1.149	0.921	0.767
6	5.532	4.099	3.008	2.296	1.841	1.535
	4.699	3.361	2.423	1.841	1.475	1.230
	4.000	2.775	1.974	1.496	1.198	0.999
	3.230	2.165	1.522	1.150	0.921	0.768
7	5.850	4.211	3.032	2.299	1.841	1.535
	4.948	3.438	2.437	1.843	1.475	1.230
	4.197	2.828	1.983	1.497	1.197	0.998
	3.376	2.200	1.527	1.150	0.920	0.766
8	6.127	4.296	3.046	2.300	1.841	1.535
	5.164	3.495	2.445	1.843	1.474	1.230
	4.307	2.868	1.988	1.497	1.197	0.999
	3.500	2.225	1.530	1.150	0.920	0.768
9	6.372	4.361	3.055	2.301	1.840	1.535
	5.354	3.538	2.450	1.843	1.474	1.230
	4.515	2.897	1.992	1.497	1.197	0.999
	3.608	2.243	1.532	1.151	0.920	0.768
10	6.590	4.412	3.060	2.301	1.839	1.535
	5.522	3.571	2.453	1.843	1.472	1.230
	4.645	2.919	1.994	1.496	1.198	0.999
	3.702	2.257	1.533	1.151	0.920	0.768
11	6.783	4.451	3.064	2.301	1.837	1.535
	5.669	3.596	2.455	1.843	1.472	1.230
	4.759	2.936	1.955	1.497	1.198	0.999
	3.784	2.267	1.534	1.151	0.920	0.765
12	6.956	4.482	3.065	2.299	1.837	1.535
	5.802	3.615	2.456	1.843	1.471	1.230
	4.862	2.949	1.994	1.497	1.198	0.999
	3.857	2.274	1.533	1.151	0.918	0.756
∞	9.210	4.605	3.070	2.302	1.842	1.535
	7.378	3.689	2.460	1.844	1.476	1.230
	5.991	2.996	1.997	1.498	1.198	0.999
	4.605	2.303	1.535	1.152	0.922	0.768

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